The Square of Opposition in Orthomodular Logic

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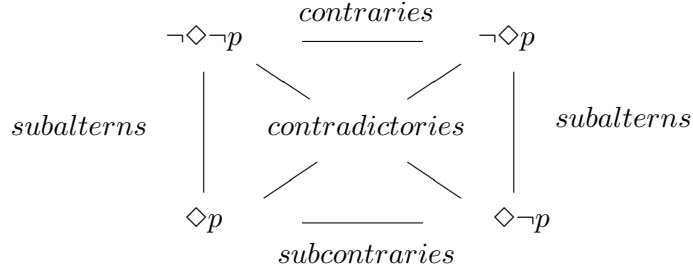Casilla de Correo 67, Sucursal 28, 1428 Buenos Aires, Argentina

Abstract

We give an interpretation of the Square of Opposition in an orthomodular structure enriched with a monadic quantifier.

Key words: square of opposition, modal orthomodular logic, classical consequences.

Introduction

In Aristotelian logic, categorical propositions are divided into four basic types: *Universal Affirmative*, *Universal Negative*, *Particular Affirmative* and *Particular Negative*. The possible relations between each two of the mentioned propositions are encoded in the famous *Square of Opposition*. The square expresses the essential properties of the monadic first order quantifiers $\forall$, $\exists$. In an algebraic approach, these properties can be represented within the frame of monadic Boolean algebras [8]. More precisely, quantifiers are considered as modal operators acting on a Boolean algebra while the Square of Opposition is represented by relations between certain terms of the language in which the algebraic structure is formulated. This representation is sometimes called *Modal Square of Opposition* and is pictured as follows:

$$\begin{array}{ccc}
 & \textit{contraries} & \\
\neg\Diamond\neg p & \overline{} & \neg\Diamond p \\
\mid & \textit{contradictories} & \mid \\
\textit{subalterns} & & \textit{subalterns} \\
\mid & & \mid \\
\Diamond p & \overline{} & \Diamond\neg p \\
 & \textit{subcontraries} &
\end{array}$$

The interpretations given to $\Diamond$ from different modal logics determine the corresponding versions of the modal Square of Opposition. By changing the underlying Boolean structure we obtain several generalizations of the monadic first order logic (see for example [9]). In turn, these generalizations give rise to new interpretations of the Square.

The aim of this paper is to study the Square of Opposition in an orthomodular structure enriched with a monadic quantifier known as *Boolean saturated orthomodular lattice* [5]. The paper is structured as follows. Section 1 contains generalities on orthomodular lattices. In Section 2, the physical motivation for the modal enrichment of the orthomodular structure is presented. In Section 3 we formalize the concept of classical consequence with respect to a property of a quantum system. Finally, in Section 4, logical relationships between the propositions embodied in a square diagram are studied in terms of classical consequences and contextual valuations.

1   Basic Notions

We recall from [1], [12] and [13] some notions of universal algebra and lattice theory that will play an important role in what follows. Let $\mathcal{L} = \langle \mathcal{L}, \vee, \wedge, 0, 1 \rangle$ be a bounded lattice. An element $c \in \mathcal{L}$ is said to be a *complement* of a iff $a \wedge c = 0$ and $a \vee c = 1$. Given a, b, c in $\mathcal{L}$, we write: $(a, b, c)D$ iff $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$; $(a, b, c)D^*$ iff $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ and $(a, b, c)T$ iff $(a, b, c)D$, (a,b,c)D^* hold for all permutations of a, b, c. An element z of a lattice $\mathcal{L}$ is called *central* iff for all elements $a, b \in \mathcal{L}$ we have $(a, b, z)T$ and z is complemented. We denote by $Z(\mathcal{L})$ the set of all central elements of $\mathcal{L}$ and it is called the *center* of $\mathcal{L}$.

A *lattice with involution* [11] is an algebra $\langle \mathcal{L}, \vee, \wedge, \neg \rangle$ such that $\langle \mathcal{L}, \vee, \wedge \rangle$ is a lattice and $\neg$ is a unary operation on $\mathcal{L}$ that fulfills the following conditions: $\neg\neg x = x$ and $\neg(x \vee y) = \neg x \wedge \neg y$. An *orthomodular lattice* is an algebra $\langle \mathcal{L}, \wedge, \vee, \neg, 0, 1 \rangle$ of type $\langle 2, 2, 1, 0, 0 \rangle$ that satisfies the following

conditions

1. $\langle \mathcal{L}, \wedge, \vee, \neg, 0, 1 \rangle$ is a bounded lattice with involution,

2. $x \wedge \neg x = 0$.

3. $x \vee (\neg x \wedge (x \vee y)) = x \vee y$

We denote by $\mathcal{OML}$ the variety of orthomodular lattices. Let L be an orthomodular lattice and $a, b \in L$. Then a commutes with b if and only if $a = (a \wedge b) \vee (a \wedge \neg b)$. A non-empty subset A is called a *Greechie set* iff for any three different elements of A, at least one of them commutes with the other two. If A is a Greechie set in L then $\langle A \rangle_L$, i.e. the sublattice generated by A, is distributive [7]. *Boolean algebras* are orthomodular lattices satisfying the *distributive law* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. We denote by **2** the Boolean algebra of two elements. Let A be a Boolean algebra. Then, as a consequence of the application of the *maximal filter theorem* for Boolean algebras, there always exists a Boolean homomorphism $f : A \to \mathbf{2}$. If $\mathcal{L}$ is a bounded lattice then $Z(\mathcal{L})$ is a Boolean sublattice of $\mathcal{L}$ [13, Theorem 4.15].

2   Modal Propositions about Quantum Systems

In the usual terms of quantum logic [2, 10], a property of a system is related to a subspace of the Hilbert space $\mathcal{H}$ of its (pure) states or, analogously, to the projector operator onto that subspace. A physical magnitude $\mathcal{M}$ is represented by an operator $\mathbf{M}$ acting over the state space. For bounded self-adjoint operators, conditions for the existence of the spectral decomposition $\mathbf{M} = \sum_i a_i \mathbf{P}_i = \sum_i a_i |a_i >< a_i|$ are satisfied. The real numbers a_i are related to the outcomes of measurements of the magnitude $\mathcal{M}$ and projectors $|a_i >< a_i|$ to the mentioned properties. Thus, the physical properties of the system are organized in the lattice of closed subspaces $\mathcal{L}(\mathcal{H})$. Moreover, each self-adjoint operator $\mathbf{M}$ has associated a Boolean sublattice $W_{\mathbf{M}}$ of $L(\mathcal{H})$ which we will refer to as the spectral algebra of the operator $\mathbf{M}$. More precisely, the family $\{\mathbf{P_i}\}$ of projector operators is identified as elements of $W_{\mathbf{M}}$. Assigning values to a physical quantity $\mathcal{M}$ is equivalent to establishing a Boolean homomorphism $v : W_{\mathbf{M}} \to \mathbf{2}$ which we call *contextual valuation*. Thus, we can say that it makes sense to use the "classical discourse" —this is, the classical logical laws are valid— within the context given by $\mathbf{M}$.

Modal interpretations of quantum mechanics [3, 4, 15] face the problem of finding an objective reading of the accepted mathematical formalism of

the theory, a reading "in terms of properties possessed by physical systems, independently of consciousness and measurements (in the sense of human interventions)"[4]. These interpretations intend to consistently include the possible properties of the system in the discourse establishing a new link between the state of the system and the probabilistic character of its properties, namely, sustaining that the interpretation of the quantum state must contain a modal aspect. The name modal interpretation was used for the first time by B. van Fraassen [14] following modal logic, precisely the logic that deals with *possibility* and *necessity*. The fundamental idea is to interpret "the formalism as providing information about properties of physical systems". A physical property of a system is "a definite value of a physical quantity belonging to this system; i.e., a feature of physical reality" [3] and not a mere measurement outcome. As usual, definite values of physical magnitudes correspond to yes/no propositions represented by orthogonal projection operators acting on vectors belonging to the Hilbert space of the (pure) states of the system [10].

The proposed modal system [5, 6] over which we now develop an interpretation of the Square of Opposition, is based on the study of the "*classical consequences*" that result from assigning values to a physical quantity. In precise terms, we enriched the orthomodular structure with a modal operator taking into account the following considerations:

1) Propositions about the properties of the physical system are interpreted in the orthomodular lattice of closed subspaces of $\mathcal{H}$. Thus, we retain this structure in our extension.

2) Given a proposition about the system, it is possible to define a context from which one can predicate with certainty about it together with a set of propositions that are compatible with it and, at the same time, predicate probabilities about the other ones (Born rule). In other words, one may predicate truth or falsity of all possibilities at the same time, i.e. possibilities allow an interpretation in a Boolean algebra. In rigorous terms, for each proposition p, if we refer with $\diamond p$ to the possibility of p, then $\diamond p$ will be a central element of a orthomodular structure.

3) If p is a proposition about the system and p occurs, then it is trivially possible that p occurs. This is expressed as $p \leq \diamond p$.

4) Let p be a property appertaining to a context $\mathcal{M}$. Assuming that

p is an actual property (for example the result of a filtering measurement) we may derive from it a set of propositions (perhaps not all of them encoded in the original Hilbert lattice of the system) which we call *classical consequences*. For example, let q be another property of the system and assign to q the probability $prob(q) = r$ via the Born rule. Then equality $prob(q) = r$ will be considered as a classical consequence of p. In fact, the main characteristic of this type of classical consequences is that it is possible to simultaneously predicate the truth of all of them (and the falsity of their negations) whenever p is true. The formal representation of the concept of classical consequence is the following: A proposition t is a classical consequence of p iff t is in the center of an orthomodular lattice containing p and satisfies the property $p \leq t$. These classical consequences are the same ones as those which would be obtained by considering the original actual property p as a possible one $\Diamond p$. Consequently $\Diamond p$ must precede all classical consequences of p. This is interpreted in the following way: $\Diamond p$ is the smallest central element greater than p.

From consideration 1, it follows that the original orthomodular structure is maintained. The other considerations are satisfied if we consider a modal operator $\Diamond$ over an orthomodular lattice $\mathcal{L}$ defined as

$$\Diamond a = Min\{z \in Z(\mathcal{L}) : a \leq z\}$$

with $Z(\mathcal{L})$ the center of $\mathcal{L}$. When this minimum exists for each $a \in \mathcal{L}$ we say that $\mathcal{L}$ is a *Boolean saturated orthomodular lattice*. We have shown that this enriched orthomodular structure can be axiomatized by equations conforming a variety denoted by $\mathcal{OML}^\Diamond$ [5]. More precisely, each element of $\mathcal{OML}^\Diamond$ is an algebra $\langle \mathcal{L}, \wedge, \vee, \neg, \Diamond, 0, 1 \rangle$ of type $\langle 2, 2, 1, 1, 0, 0 \rangle$ such that $\langle \mathcal{L}, \wedge, \vee, \neg, 0, 1 \rangle$ is an orthomodular lattice and $\Box$ satisfies the following equations:

S1 $\quad x \leq \Diamond x$ $\qquad\qquad$ S5 $\quad y = (y \wedge \Diamond x) \vee (y \wedge \neg \Diamond x)$

S2 $\quad \Diamond 0 = 0$ $\qquad\qquad$ S6 $\quad \Diamond(x \wedge \Diamond y) = \Diamond x \wedge \Diamond y$

S3 $\quad \Diamond \Diamond x = \Diamond x$ $\qquad\quad$ S7 $\quad \neg \Diamond x \wedge \Diamond y \leq \Diamond(\neg x \wedge (y \vee x))$

S4 $\quad \Diamond(x \vee y) = \Diamond x \vee \Diamond y$

Orthomodular complete lattices are examples of Boolean saturated orthomodular lattices. We can embed each orthomodular lattice $\mathcal{L}$ in an element $\mathcal{L}^\Diamond \in \mathcal{OML}^\Diamond$ see [5, Theorem 10]. In general, $\mathcal{L}^\Diamond$ is referred as a

modal extension of $\mathcal{L}$. In this case we may see the lattice $\mathcal{L}$ as a subset of $\mathcal{L}^\diamond$.

3 Modal Extensions and Classical Consequences

We begin our study of the Square of Opposition analyzing the classical consequences that can be derived from a proposition about the system. This idea was suggested in the condition 4 of the motivation of the structure $\mathcal{OML}^\diamond$. In what follows we express the notion of classical consequence as a formal concept in $\mathcal{OML}^\diamond$. We first need the following technical results:

Proposition 3.1 *Let L be an orthomodular lattice. If A is a Greechie set in L such that for each $a \in A, \neg a \in A$ then, $\langle A \rangle_L$ is Boolean sublattice.*

Proof: It is well known from [7] that $\langle A \rangle_L$ is a distributive sublattice of L. Since distributive orthomodular lattices are Boolean algebras, we only need to see that $\langle A \rangle_L$ is closed by $\neg$. To do that we use induction on the complexity of terms of the subuniverse generated by A. For $comp(a) = 0$, it follows from the fact that A is closed by negation. Assume validity for terms of the complexity less than n. Let τ be a term such that $comp(\tau) = n$. If $\tau = \neg \tau_1$ then $\neg \tau \in \langle A \rangle_L$ since $\neg \tau = \neg \neg \tau_1 = \tau_1$ and $\tau_1 \in \langle A \rangle_L$. If $\tau = \tau_1 \wedge \tau_2$, $\neg \tau = \neg \tau_1 \vee \neg \tau_2$. Since $comp(\tau_i) < n$, $\neg \tau_i \in \langle A \rangle_L$ for $i = 1, 2$ resulting $\neg \tau \in \langle A \rangle_L$. We use the same argument in the case $\tau = \tau_1 \vee \tau_2$. Finally $\langle A \rangle_L$ is a Boolean sublattice.

$\square$

Since the center $Z(\mathcal{L}^\diamond)$ is a Boolean algebra, it represents a fragment of discourse added to $\mathcal{L}$ in which the laws of the classical logic are valid. Thus the modal extension $\mathcal{L} \hookrightarrow \mathcal{L}^\diamond$ is a structure that rules the mentioned fragment of classical discourse and the properties about a quantum system encoded in $\mathcal{L}$. Let W be a Boolean sub-algebra of $\mathcal{L}$ (i.e. a context). Note that $W \cup Z(\mathcal{L}^\diamond)$ is a Greechie set closed by $\neg$. Then by Proposition 3.1, $\langle W \cup Z(\mathcal{L}^\diamond)\rangle_{\mathcal{L}^\diamond}$ is a Boolean sub-algebra of $Z(\mathcal{L}^\diamond)$. This represents the possibility to fix a context and compatibly add a fragment of classical discourse. We will refer to the Boolean algebra $W^\diamond = \langle W \cup Z(\mathcal{L}^\diamond)\rangle_{\mathcal{L}^\diamond}$ as a *classically expanded context*. Taking into account that assigning values to a physical quantity p is equivalent to fix a context W in which $p \in W$ and establish a Boolean homomorphism $v : W \to \mathbf{2}$ such that $v(p) = 1$, we give the following definition of classical consequence.

Definition 3.2 Let $\mathcal{L}$ be an orthomodular lattice, $p \in \mathcal{L}$ and $\mathcal{L}^{\diamondsuit} \in \mathcal{OML}^{\diamondsuit}$ a modal extension of $\mathcal{L}$. Then $z \in Z(\mathcal{L}^{\diamondsuit})$ is said to be a *classical consequence* of p iff for each Boolean sublattice W in $\mathcal{L}$ (with $p \in W$) and each Boolean valuation $v : W^{\diamondsuit} \to \mathbf{2}$, $v(z) = 1$ whenever $v(p) = 1$.

$v(p) = 1$ implies $v(z) = 1$ in Definition 3.2 is a relation in a classically expanded context that represents, in an algebraic way, the usual logical consequence of z from p. We denote by $Cons_{\mathcal{L}^{\diamondsuit}}(p)$ the set of classical consequences of p in the modal extension $\mathcal{L}^{\diamondsuit}$.

Proposition 3.3 *Let $\mathcal{L}$ be an orthomodular lattice, $p \in \mathcal{L}$ and $\mathcal{L}^{\diamondsuit} \in \mathcal{OML}^{\diamondsuit}$ a modal extension of $\mathcal{L}$. Then we have that*

$$Cons_{\mathcal{L}^{\diamondsuit}}(p) = \{z \in Z(\mathcal{L}^{\diamondsuit}) : p \leq z\} = \{x \in Z(\mathcal{L}^{\diamondsuit}) : \diamondsuit p \leq z\}$$

Proof: $\{z \in Z(\mathcal{L}^{\diamondsuit}) : p \leq z\} = \{z \in Z(\mathcal{L}^{\diamondsuit}) : \diamondsuit p \leq z\}$ follows from definition of $\diamondsuit$. The inclusion $\{z \in Z(\mathcal{L}^{\diamondsuit}) : \diamondsuit p \leq z\} \subseteq Cons_{\mathcal{L}^{\diamondsuit}}(p)$ is trivial. Let $z \in Cons_{\mathcal{L}^{\diamondsuit}}(p)$ and suppose that $p \nleq z$. Consider the Boolean sub-algebra of $\mathcal{L}$ given by $W = \{p, \neg p, 0, 1\}$. By the maximal filter theorem, there exists a maximal filter F in $W^{\diamondsuit}$ such that $p \in F$ and $z \notin F$. If we consider the quotient Boolean algebra $W^{\diamondsuit}/F = \mathbf{2}$, the natural Boolean homomorphism $f : W^{\diamondsuit} \to \mathbf{2}$ satisfies that $f(p) = 1$ and $f(z) = 0$, which is a contradiction. Hence $p \nleq z$ and $z \in Cons_{\mathcal{L}^{\diamondsuit}}(p)$.

$\square$

The equality $Cons_{\mathcal{L}^{\diamondsuit}}(p) = \{x \in Z(\mathcal{L}^{\diamondsuit}) : \diamondsuit p \leq z\}$ given in Proposition 3.3 states that the notion of classical consequence of p results independent of the choice of the context W in which $p \in W$. In fact, each possible classical consequence $z \in Z(\mathcal{L}^{\diamondsuit})$ of p is only determined by the relation $\diamondsuit p \leq z$. Thus $Z(\mathcal{L}^{\diamondsuit})$ is a fragment of the classical discourse added to $\mathcal{L}$ which allows to "predicate" classical consequences about the properties of the system encoded in $\mathcal{L}$ independently of the context. It is important to remark that the contextual character of the quantum discourse is only avoided when we refer to "classical consequences of properties about the system" and not when referring to the properties in themselves, i.e. independently of the choice of the context. In fact, in our modal extension, the discourse about properties is genuinely enlarged, but the contextual character remains a main feature of quantum systems even when modalities are taken into account.

4 Square of Opposition: Classical Consequences and Contextual Valuations

In this section we analyze the relations between propositions encoded in the scheme of the Square of Opposition in terms of the classical consequences of a chosen property about the quantum system. To do this, we use the modal extension. Let $\mathcal{L}$ be an orthomodular lattice, $p \in \mathcal{L}$ and $\mathcal{L}^{\diamond} \in \mathcal{OML}^{\diamond}$ a modal extension of $\mathcal{L}$. We first study the proposition $\neg\diamond\neg p$ denoted by $\square p$. Note that $\square p = \neg\diamond\neg p = \neg Min\{z \in Z(\mathcal{L}^{\diamond}) : \neg p \leq z\} = Max\{\neg z \in Z(\mathcal{L}^{\diamond}) : \neg p \leq z\} = Max\{\neg z \in Z(\mathcal{L}^{\diamond}) : \neg z \leq p\}$. Considering $t = \neg z$ we have that

$$\square p = \neg\diamond\neg p = Max\{t \in Z(\mathcal{L}^{\diamond}) : t \leq p\}$$

When W is a Boolean sublattice of $\mathcal{L}$ such that $p \in W$ (i.e. we are fixing a context containing p), $\square p$ is the greatest classical proposition that implies p in the classically expanded context $W^{\diamond}$. More precisely, if p is a consequence of $z \in \mathcal{L}^{\diamond}$ then $\square p$ is consequence of z.

Remark 4.1 It is important to notice that Proposition 3.3 allows us to refer to classical consequences of a property of the system independently of the chosen context. But in order to refer to a property which is implied by a classical property, we need to fix a context and consider the classically expanded context.

Lemma 4.2 *Let $\mathcal{L}$ be an orthomodular lattice, $p \in \mathcal{L}$ and $\mathcal{L}^{\diamond}$ be a modal extension of $\mathcal{L}$. If $\diamond p \wedge \diamond\neg p = 0$ then $p \in Z(\mathcal{L})$.*

Proof: $\diamond p$ and $\diamond\neg p$ are central elements in $\mathcal{L}^{\diamond}$. Taking into account that $1 = p \vee \neg p \leq \diamond p \vee \diamond\neg p$, if $\diamond p \wedge \diamond\neg p = 0$ then $\neg\diamond p = \diamond\neg p$ since the complement is unique in $Z(\mathcal{L}^{\diamond})$. Hence $\square p = \neg\diamond\neg p = \diamond p$, $p \in Z(\mathcal{L}^{\diamond})$ and $p \in Z(\mathcal{L})$.

$\square$

Now we can interpret the relation between propositions in the Square of Opposition. In what follows we assume that $\mathcal{L}$ is an orthomodular lattice and $p \in \mathcal{L}$ such that $p \notin Z(\mathcal{L})$, i.e. p is not a classical proposition in a quantum system represented by $\mathcal{L}$. Let $\mathcal{L}^{\diamond}$ be a modal extension of $\mathcal{L}$, W be a Boolean subalgebra of $\mathcal{L}$, i.e. a context, such that $p \in W$ and consider a classically expanded context $W^{\diamond}$.

- $\neg\diamond\neg p$ *contraries* $\neg\diamond p$

$\neg \Diamond p = \neg \Diamond \neg \neg p = \Box \neg p$. Thus, the *contrary proposition* is the greatest classical proposition that implies p, i.e. $\Box p$, with respect to the greatest classical proposition that implies $\neg p$, i.e. $\Box \neg p$, in each possible classically expanded context containing $p, \neg p$.

In the usual explanation, two propositions are contrary iff they cannot both be true but can both be false. In our framework we can obtain a similar concept of contrary propositions. Note that $\Box p \wedge \Box \neg p \leq p \wedge \neg p = 0$. Thus, there is not a maximal Boolean filter containing $\Box p$ and $\Box \neg p$. Hence there is not a Boolean valuation $v : W^\Diamond \to \mathbf{2}$ such that $v(\Box p) = v(\Box \neg p) = 1$, i.e. $\Box p$ and $\Box \neg p$ "cannot both be true" in each possible classically expanded context.

Since $p \notin Z(\mathcal{L})$, by Lemma 4.2, $\Diamond p \wedge \Diamond \neg p \neq 0$. Then there exists a maximal Boolean filter F in $W^\Diamond$ containing $\Diamond p$ and $\Diamond \neg p$. $\Box p \notin F$ otherwise $\Box p \wedge \Diamond \neg p \in F$ and $\Box p \wedge \Diamond \neg p = \Diamond(\Box p \wedge \neg p) \leq \Diamond(p \wedge \neg p) = 0$ which is a contradiction. With the same argument we can prove that $\Box \neg p \notin F$. If we consider the natural homomorphism $v : W^\Diamond \to W^\Diamond/F \approx \mathbf{2}$ then $v(\Box p) = v(\Box \neg p) = 0$, i.e. $\Box p$ and $\Box \neg p$ can both be false.

- $\Diamond p$ <u>*subcontraries*</u> $\Diamond \neg p$

The *sub-contrary proposition* is the smallest classical consequence of p with respect to the smallest classical consequence of $\neg p$. Note that sub-contrary propositions do not depend on the context.

In the usual explanation, two propositions are sub-contrary iff they cannot both be false but can both be true. Suppose that there exists a Boolean homomorphism $v : W^\Diamond \to \mathbf{2}$ such that $v(\Diamond p) = v(\Diamond \neg p) = 0$. Consider the maximal Boolean filter given by $Ker(v)$. Since $Ker(v)$ is a maximal filter in $W^\Diamond$, $p \in F_v$ or $\neg p \in F_v$. If $p \in F_v$ then $v(\Diamond p) = 1$ which is a contradiction, if $\neg p \in F_v$ then $v(\Diamond \neg p) = 1$ which is a contradiction too. Hence $v(\Diamond p) \neq 0$ or $v(\Diamond \neg p) \neq 0$, i.e. they cannot both be false. Since $p \notin Z(\mathcal{L})$, by Lemma 4.2, $\Diamond p \wedge \Diamond \neg p \neq 0$. Then there exists a maximal Boolean filter F in $W^\Diamond$ containing $\Diamond p$ and $\Diamond \neg p$. Hence the Boolean homomorphism $v : W^\Diamond \to W^\Diamond/F \approx \mathbf{2}$ satisfies that $v(\Diamond p) = v(\Diamond \neg p) = 1$, i.e. $\Diamond p$ and $\Diamond \neg p$ can both be true.

- $\neg \Diamond \neg p$ <u>*subalterns*</u> $\Diamond p$ and $\neg \Diamond p$ <u>*subalterns*</u> $\Diamond \neg p$

The notion of sub-contrary propositions is reduced to the relation between $\Box p$ and $\Diamond p$. The *subaltern proposition* is the greatest classical proposition that implies p with respect to the smallest classical consequence of p.

In the usual explanation, a proposition is subaltern of another one called *superaltern*, iff it must be true when its superaltern is true, and the superaltern must be false when the subaltern is false. In our case $\neg\Diamond\neg p = \Box p$ is superaltern of $\Diamond p$ and $\neg\Diamond p = \Box\neg p$ is superaltern of $\Diamond\neg p$. Since $\Box p \leq p \leq \Diamond p$, for each valuation $v : W^\Diamond \to \mathbf{2}$, if $v(\Box p) = 1$ then $v(\Diamond p) = 1$ and if $v(\Diamond p) = 0$ then $v(\Box p) = 0$.

- $\neg\Diamond\neg p$ $\underline{contradictories}$ $\Diamond\neg p$ and $\Diamond p$ $\underline{contradictories}$ $\neg\Diamond p$

The notion of *contradictory proposition* can be reduced to the relation between $\Diamond p$ and $\Box\neg p$. The contradictory proposition is the greatest classical proposition that implies $\neg p$ with respect to the the smallest classical consequence of p. In the usual explanation, two propositions are contradictory iff they cannot both be true and they cannot both be false. Due to the fact that $ker(v)$ is a maximal filter in $W^\Diamond$, each maximal filter F in $W^\Diamond$ contains exactly one of $\{\Diamond p, \neg\Diamond p\}$ for each Boolean homomorphism $v : W^\Diamond \to \mathbf{2}$, $v(\Diamond p) = 1$ and $v(\neg\Diamond p) = 0$ or $v(\Diamond p) = 0$ and $v(\neg\Diamond p) = 1$. Hence, $\Diamond p$ and $\Diamond p$ cannot both be true and they cannot both be false.

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