An Elementary Square of Opposition at the Basis of Analogical and other Related Proportions

Henri Prade - Gilles Richard

IRIT, Toulouse, France
Reasoning (in Greek terms)

- **top-down** forms of aiming at cataloguing items by classifying them into categories and subcategories

- **down-top** forms aiming at analogizing, i.e., comparing particular items.
Analogical proportion

“a is to b as c is to d” denoted $a : b :: c : d$

Case-based reasoning:

- a problem
- b new problem
- c solution(a)
- d solution(b) ???

a, b, c: known - d (partially) unknown

“intelligence” test

- [Diagram of shapes: square, circle, triangle, question mark]
“intelligence” test (2)

Thomas G. Evans, 1964

Formal studies in the last 10 years:
Yves Lepage, Nicolas Stroppa & François Yvon, Laurent Miclet & Arnaud Delhay

... but there was no logical modelling
Contents

- The square of basic comparisons
- The logical modeling of analogical proportions
- Other logical proportions
Elementary analysis ...

2 objects $a$, $b$:

$a$ is to $b$

$a : (\text{sim}(a, b), \text{dif}(a, b))$

- *similarity*: $\text{sim}(a, b)$
- *difference*: $\text{dif}(a, b)$

- $\text{sim}$ symmetrical $\text{sim}(a,b) = \text{sim}(b,a)$
- $\text{dif}$ is not $\text{dif}(a, b) \neq \text{dif}(b, a)$
Comparing $a$ and $b$ (as sets of features)

Similarity:

$$s_1 = a \cap b \text{ and } s_2 = a^c \cap b^c$$

Dissimilarity:

$$d_1 = a \cap b \text{ and } d_2 = a^c \cap b$$

\[
\begin{align*}
\text{s1} &= a \cap b \\
\text{d1} &= a \cap b^c \\
\text{d2} &= a^c \cap b \\
\text{s2} &= a^c \cap b^c
\end{align*}
\]
Comparing $a$ and $b$ (in propositional terms)

Similarity: conjunctions

\[ s_1 = a \land b \quad \text{and} \quad s_2 = \neg a \land \neg b \]

Dissimilarity: conjunctions

\[ d_1 = a \land \neg b \quad \text{and} \quad d_2 = \neg a \land b \]

\[
\begin{array}{c}
| s_1 = a \land b \quad \text{------------------} \quad d_1 = a \land \neg b \\
| | \quad \text{------------------} \quad | \\
| | d_2 = \neg a \land b \quad \text{------------------} \quad s_2 = \neg a \land \neg b \\
\end{array}
\]
“a is to b as c is to d”

4 items a, b and c, d:

• 2 options “… as …”

• Give priority to Similarity

• Give priority to Difference (dissimilarity)
3 informal definitions ...

Difference :
1. $\text{dif}(a, b) = \text{dif}(c, d)$ \ et \ $\text{dif}(b, a) = \text{dif}(d, c)$

or
2. $\text{dif}(a, b) = \text{dif}(d, c)$ \ et \ $\text{dif}(b, a) = \text{dif}(c, d)$

Similarity :
3. $\text{sim}(a, b) = \text{sim}(c, d)$

3 proportions:
1. Analogy
2. Inverse Analogy
3. Paralogy
   “what a and b have in common, c and d have it also”
Set Interpretation

a, b, c, d: sets of binary features

a : b :: c : d iff
\[ a \cap b^c = c \cap d^c \quad \text{and} \quad b \cap a^c = d \cap c^c \]

a ; b ;; c ; d iff
\[ a \cap b^c = d \cap c^c \quad \text{and} \quad b \cap a^c = c \cap d^c \]

a ! b !! c ! d iff
\[ a \cap b = c \cap d \quad \text{and} \quad a^c \cap b^c = c^c \cap d^c \]
\[ \text{or} \quad a \cap b = c \cap d \quad \text{and} \quad a \cup b = c \cup d \]
3 sets of postulates ...

• **Analogy**
  
  \[ a : b :: a : b \text{ (and } a : a :: b : b) \text{ but not } (a : b :: b : a) \]
  
  \[ a : b :: c : d \text{ entails } c : d :: a : b \text{ (symmetry)} \]
  
  \[ a : b :: c : d \text{ entails } a : c :: b : d \text{ (central permutation)} \]

• **Inverse Analogy**
  
  \[ a ; b ;; b ; a \text{ (and } a ; a ;; b ; b) \text{ but not } (a ; b ;; a ; b) \]
  
  \[ a ; b ;; c ; d \text{ entails } c ; d ;; a ; b \text{ (symmetry)} \]
  
  \[ a ; b ;; c ; d \text{ entails } c ; b ;; a ; d \text{ (odd permutation)} \]

• **Paralogy** (or **parallelogy**)
  
  \[ a ! b !! a ! b \text{ (and } a ! b !! b ! a) \text{ but not } (a ! a !! b ! b) \]
  
  \[ a ! b !! c ! d \text{ entails } c ! d !! a ! b \text{ (symmetry)} \]
  
  \[ a ! b !! c ! d \text{ entails } b ! a !! c ! d \text{ (even permutation)} \]
Some properties ...

• Transitivity: NO !

• Central permutation: NO for inverse analogy and paralogy

• 3 Classes of 8 valid permutations

\[ a : b :: c : d \iff b : a :: d : c \]

• \[ a : b :: c : d \iff a ! b !! d ! c \iff a ; d ; ; c ; b \]
Logical Interpretation

\[
\text{a} : \text{b} :: \text{c} : \text{d} \iff ((\text{a} \rightarrow \text{b} \equiv \text{c} \rightarrow \text{d}) \land (\text{b} \rightarrow \text{a} \equiv \text{d} \rightarrow \text{c}))
\]

\[
\text{a} \!\! \text{b} \!\! \text{c} \!\! \text{d} \iff ((\text{a} \rightarrow \text{b} \equiv \text{d} \rightarrow \text{c}) \land (\text{b} \rightarrow \text{a} \equiv \text{c} \rightarrow \text{d}))
\]

\[
\text{a} ; \text{b} ; ; \text{c} ; \text{d} \iff ((\text{a} \land \text{b} \equiv \text{c} \land \text{d}) \land (\text{a} \lor \text{b} \equiv \text{c} \lor \text{d}))
\]

<table>
<thead>
<tr>
<th>Analogy</th>
<th>Reverse</th>
<th>Paralogy</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0 0 0</td>
<td>0 0 0 0</td>
<td>0 0 0 0</td>
</tr>
<tr>
<td>1 1 1 1</td>
<td>1 1 1 1</td>
<td>1 1 1 1</td>
</tr>
<tr>
<td>0 0 1 1</td>
<td>0 0 1 1</td>
<td>1 0 0 1</td>
</tr>
<tr>
<td>1 1 0 0</td>
<td>1 1 0 0</td>
<td>0 1 1 0</td>
</tr>
<tr>
<td>0 1 0 1</td>
<td>0 1 1 0</td>
<td>0 1 0 1</td>
</tr>
<tr>
<td>1 0 1 0</td>
<td>1 0 0 1</td>
<td>1 0 1 0</td>
</tr>
</tbody>
</table>

\[
\text{a} : \text{b} :: \neg \text{b} : \neg \text{a}
\]
### Intelligence test (continued)

<table>
<thead>
<tr>
<th></th>
<th>square</th>
<th>blue cercle</th>
<th>green</th>
<th>yellow</th>
<th>triangle</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>d</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

*Images of shapes are shown in the text.*
The 3 proportions

Fig. 1. Graphical analogy

Fig. 2. Graphical reverse analogy

Fig. 3. Graphical paralogy
(Other) Logical Proportions

Def. A *logical proportion* is defined via a *pair* of equalities of the form

\[
\alpha \cap \beta = \gamma \cap \delta \\
\alpha' \cap \beta' = \gamma' \cap \delta'
\]

\[\alpha \in \{a, \neg a\}, \beta \in \{b, \neg b\}, \ldots\]

\[\alpha' \in \{a, \neg a\}, \beta' \in \{b, \neg b\}, \ldots\]

where

- i) the equalities are distinct
- ii) their ordering is irrelevant
How many logical proportions?

\[ C^6_{16} = 8008 ? \]  No!  only 120

similarities: \( s_1 = a \cap b \), \( s_2 = \neg a \cap \neg b \),
\( s'_1 = c \cap d \), \( s'_2 = \neg c \cap \neg d \)
dissimilarities: \( d_1 = a \cap \neg b \), \( d_2 = \neg a \cap b \),
\( d'_1 = c \cap \neg d \), \( d'_2 = \neg c \cap d \)

- 4 homogeneous: 2 cond. \( s_i = s'_k \) or 2 cond. \( d_i = d'_k \)
- 3 + reversed paralogy: \( a \cap b = \neg c \cap \neg d \) and \( \neg a \cap \neg b = c \cap d \)
- 16 conditionals: \( s_i = s'_k \) and \( d_j = d'_l \)
- 20 hybrids: \( s_i = d'_k \) and \( s_j = d'_l \)
- 32 semi-hybrids: \( s_i = s'_k \) or \( d_j = d'_l \) + 1 hybrid cond.
- 48 degenerated: the same \( s_i \) (or \( s'_k \), \( d_j \), \( d'_l \)) in the 2 cond.
The 4 homogeneous proportions

Analogy
(d₁≡d’₁ ∧ d₂≡d’₂)

Paralogy
(s₁≡s’₁ ∧ s₂≡s’₂)

Inverse analogy
(d₁≡d’₂ ∧ d₂≡d’₁)

Reverse paralogy
(s₁≡s’₂ ∧ s₂≡s’₁)

the primed symbols refer to c and d
Desirable properties (?)

Proportion $T$

**Full identity**: $T(a, a, a, a)$

Identity: $T(a, a, b, b)$

Reflexivity: $T(a, b, a, b)$

Reverse reflexivity: $T(a, b, b, a)$

**Symmetry** $T(a, b, c, d) = T(c, d, a, b)$

**Independence w.r.t. coding**: $T(a, b, c, d) = T(\neg a, \neg b, \neg c, \neg d)$
Different classes

15 proportions satisfying **full identity**
(3 homogeneous, 8 conditional, 4 degenerated)

30 proportions satisfy 1 1 1 1 but not 0 0 0 0
(4 hybrids, 12 semi-hybrids, 14 degenerated)

30 proportions satisfy 0 0 0 0 but not 1 1 1 1
(4 hybrids, 12 semi-hybrids, 14 degenerated)

45 proportions have neither 0 0 0 0 nor 1 1 1 1 in their table
(1 homogeneous (**reversed paralogy**), 8 conditionals, 12 hybrids,
  8 semi-hybrids, 16 degenerated)

12 **symmetrical** proportions:
4 homogeneous + 4 conditionals + 4 hybrids

8 proportions **independence wrt coding**:
  4 homogeneous + 4 hybrids
The 15 proportions that satisfy **full identity**

### Table 1: Analogy-related proportions

<table>
<thead>
<tr>
<th></th>
<th>Direct Analogy</th>
<th>Rev. Analogy</th>
<th>Paralogy</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$a : b :: c : d$</td>
<td>$a ! b :: c ! d$</td>
<td>$a ; b :: c ; d$</td>
</tr>
<tr>
<td>$a$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$b$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$c$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$d$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

www.irit.fr
Conditional proportions

- \( a \cap b = c \cap d \) and \( a \cap \overline{b} = c \cap \overline{d} \), denoted \( b|a :: d|c \)
- \( a \cap b = c \cap d \) and \( \overline{a} \cap b = \overline{c} \cap d \), denoted \( a|b :: c|d \)
- \( \overline{a} \cap \overline{b} = \overline{c} \cap \overline{d} \) and \( \overline{a} \cap b = \overline{c} \cap d \), denoted \( \overline{b}|\overline{a} :: \overline{d}|\overline{c} \)
- \( \overline{a} \cap \overline{b} = \overline{c} \cap \overline{d} \) and \( a \cap \overline{b} = c \cap \overline{d} \), denoted \( \overline{a}|b :: c|d \)

**Conditional objects**! (same examples, same counter-examples)

<table>
<thead>
<tr>
<th>Table 2: Direct proportions (other than analogy)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b</td>
</tr>
<tr>
<td>---------------</td>
</tr>
<tr>
<td>( a )</td>
</tr>
<tr>
<td>( 0 )</td>
</tr>
<tr>
<td>( 1 )</td>
</tr>
<tr>
<td>( 0 )</td>
</tr>
<tr>
<td>( 1 )</td>
</tr>
<tr>
<td>( 0 )</td>
</tr>
<tr>
<td>( 0 )</td>
</tr>
<tr>
<td>( 0 )</td>
</tr>
</tbody>
</table>

**Full identity + symmetry**: 7 proportions
Other conditional proportions

- \( a \cap b = c \cap d \) and \( a \cap \overline{b} = \overline{c} \cap d \), denoted \( b|a :: c|d \)
- \( a \cap b = c \cap d \) and \( \overline{a} \cap b = c \cap \overline{d} \), denoted \( a|b :: d|c \)
- \( \overline{a} \cap \overline{b} = \overline{c} \cap \overline{d} \) and \( \overline{a} \cap b = c \cap \overline{d} \), denoted \( \overline{a}|\overline{b} :: \overline{c}|\overline{d} \)
- \( \overline{a} \cap \overline{b} = \overline{c} \cap \overline{d} \) and \( a \cap \overline{b} = \overline{c} \cap d \), denoted \( a|\overline{b} :: d|\overline{c} \)

Table 3: Reverse proportions (other than rev. analogy)

<table>
<thead>
<tr>
<th>b</th>
<th>a :: c</th>
<th>d</th>
<th>a</th>
<th>b :: d</th>
<th>c</th>
<th>b</th>
<th>a :: c</th>
<th>d</th>
<th>a</th>
<th>b :: d</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
+ Basic inclusions ...

\[ a \cap b = c \cap d \quad \text{and} \quad \overline{a} \cap \overline{b} = \overline{c} \cap \overline{d} \quad (1) \]

\[ \overline{a} \cap b = \overline{c} \cap d \quad \text{and} \quad a \cap b = \overline{c} \cap d \quad (2) \]

\[ a \cap b = c \cap d \quad \text{and} \quad \overline{a} \cap \overline{b} = c \cap d \quad (3) \]

\[ a \cap b = c \cap d \quad \text{and} \quad a \cap b = \overline{c} \cap d \quad (4) \]

Table 4: The 4 generated proportions satisfying full identity

|   |   |   |   |   | a | b | c | d |   |   |   |   | a | b | c | d |   |   |   |   | a | b | c | d |
| 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 4 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
Conclusion

3 proportions linked with analogy

- Analogy
- Inverse Analogy
- Paralogy

And many other / distinct intuitions

Multiple-valued logic extensions

Applications

- Inferences based on proportions
- Classification by analogical transduction

Much remains to be done ...

- Experimentations binary and multiple-valued cases
Sheldon Klein ...

B.A. in anthropology in 1956
Ph.D. in linguistics in 1963
Died at 70, on July 22, 2005

S. Klein. **Culture, mysticism & social structure and the calculation of behavior.**

S. Klein The analogical foundations of creativity in language, culture and the arts.
2002.

http://pages.cs.wisc.edu/~sklein/sklein.html
Un extrait de S. Klein (ECAI’82) …

A 3-valued variant is useful for state transitions where events emerge that were not present in the initial state. One can represent this with the 2-valued ATO, but the 3-valued variant is also useful. Again a reversal of the interpretation of 1 & 0 yields an implementation as non-carry addition.

Complex analogies may also be computed, e.g.,
References

H. Prade et G. Richard. Logical proportions: Typology and roadmap. (IPMU’10), Dortmund

H. Prade et G. Richard. Multiple-valued logic interpretations of analogical, reverse analogical, and paralogical proportions. (ISMVL’10), Barcelona

H. Prade et G. Richard. Nonmonotonic features and uncertainty in reasoning with analogical proportions. (NMR’10), Toronto,

H. Prade et G. Richard. Reasoning with logical proportions. (KR’10), Toronto


Patterns in the 3-valued case

Ordinal scale \{0, \alpha, 1\} \quad \alpha = \neg \alpha

19 « perfect analogies »: \begin{align*}
1 : \alpha &:: 1 : \alpha \\
1 : 1 &:: \alpha : \alpha \\
\alpha : \alpha &:: \alpha : \alpha \\
1 : \alpha &:: \alpha : 0
\end{align*}

40 approximate analogies: \begin{align*}
1 : \alpha &:: 1 : 0
\end{align*}

22 not analogies: \begin{align*}
1 : 0 &:: \alpha : \alpha
\end{align*}

Similar analysis for the 2 other proportions
« Graded » intelligence tests

Fig. 4. Fully true analogy

Fig. 5. Fully false analogy
multiple-valued interpretations ... potentially many!

\[ a : b :: c : d \text{ iff } ((a \rightarrow b \equiv c \rightarrow d) \land (b \rightarrow a \equiv d \rightarrow c)) \]

\[ s \equiv t = 1 \iff s = t \]
\[ s \equiv t \text{ decreases all the more as } s \text{ and } t \text{ differ} \]
\[ s \equiv t = 1 \iff |s - t| \]
\[ s \equiv t \iff \neg s \equiv \neg t \]
\[ \land \text{ min} \]
multiple-valued interpretations ...

potentially many!

\[ a : b :: c : d \iff ((a \to b \equiv c \to d) \land (b \to a \equiv d \to c)) \]

\[ \rightarrow \quad 2 \text{ reasonable postulates} \ldots \]

i) Independence w.r.t. coding \quad (a \to b \equiv \neg b \to \neg a)

\[ + \]

ii) Reconstructibility

\[ b = a \land (a \to b) \lor \neg(b \to a) \quad (\star) \]
**multiple-valued interpretations ...**

★ holds
if \( \rightarrow \) residuated w.r.t. \( \land \), has contrapositive symmetry
(and \( \land \) is the dual of \( \lor \) w.r.t. \( \neg \))

➢ Lukasiewicz implication \( a \rightarrow b = \min(1, 1 - a + b) \)

➢ \( a : b :: c : d = 1 - l(a - b) - (c - d) l \) if \( a \geq b \) and \( c \geq d \)
   or if \( a \leq b \) and \( c \leq d \)
   \[ = 1 - \max(la - bl, lc - dl) \] otherwise

• perfectly agrees with 3-valued patterns
• *nilpotent implication* ok, but \( 1/4 : 0 :: 1/2 : 1/4 = 3/4 ! \)
Paralogy

\[ a ; b ; c ; d \text{ iff } ((a \land b \equiv c \land d) \land (a \lor b \equiv c \lor d)) \]

\[ \land \quad \text{min} \]
\[ \lor \quad \text{max} \]

i) Independence w.r.t. coding

\[ + \]

ii) Reconstructibility

\[ b = (a \land b) \lor ((a \lor b) \land \neg a) \]

perfect paralogies: only \( a ; b ; c ; b \) and \( a ; b ; b ; a \)
Post algebra

Ordered set \( \{a_1, \ldots, a_n\} \)

e.g. \( \{\text{triangle, square, hexagon, circle}\} \)

\[ \sigma(a_i) = a_{i+1} \quad \text{for} \quad i = 1, n-1; \quad \sigma(a_n) = a_1 \]

*Post’s negation:* \( \neg a = \sigma(a) \)

- Patterns \( a : \sigma^k(a) :: a : \sigma^k(a) \)
  and \( a : a :: \sigma^k(a) : \sigma^k(a) \)
A simple inference rule

\[ a, b, c, d : \quad 4 \text{ vectors of binary components} \]

All the attributes are known pour \( a, b, c \)

- Only a part for \( d \)
- A formal proportion \( P \) holds

\[
\begin{array}{c|c|c|c|c}
\text{a} & \text{b} & \text{c} & \text{d} \\
\hline
\text{known} & \text{known} & \text{P holds} & \text{known} \\
\text{P should hold !} & \text{unknown} \\
\end{array}
\]

equation solving problem...
Example...

<table>
<thead>
<tr>
<th></th>
<th>car</th>
<th>diesel</th>
<th>4mw</th>
<th>green</th>
<th>expensive</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>b</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>d</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>? (1)</td>
<td>? (0)</td>
</tr>
</tbody>
</table>
Clausal forms

Clausal form of a logical proportion

La forme clausale de la proportion analogique est :
\[
\{\neg a \lor b \lor c, \neg a \lor b \lor \neg d, a \lor \neg c \lor d, \neg b \lor \neg c \lor d,
\]
\[
a \lor \neg b \lor \neg c, a \lor \neg b \lor d, \neg a \lor c \lor \neg d, b \lor c \lor \neg d\}
\]

la paralogie est :
\[
\{\neg a \lor c \lor d, \neg a \lor \neg b \lor d, a \lor b \lor \neg c, b \lor \neg c \lor \neg d,
\]
\[
a \lor \neg c \lor \neg d, a \lor \neg b \lor \neg d, \neg a \lor \neg b \lor c, \neg b \lor c \lor d\}.\]
\[ \neg a, \quad b, \quad \neg c, \quad a : b :: c : d \]

\[ d = c \equiv (a \equiv b) \]
A tri-valued interpretation...

- $T = \{0, -1, 1\}$
- $a, b, c, d \in T \times T$ (9 elements)

\[
a - b = c - d \quad (A) \quad a - b = d - c \quad (R) \quad a + b = c + d \quad (P)
\]