# Reversed Squares of Opposition in PAL and DEL

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# Goal of the talk

- public announcement logic (PAL), and dynamic epistemic logic (DEL) in general, are recent field of epistemic logic that study how knowledge *changes* under the influence of various *actions* (e.g. communication between agents)
- many applications in *computer science* (protocol security), *AI* (multi-agent systems), *game theory* (backward induction paradox), *linguistics* (formal semantics) etc. . .
- the goal of this talk is to connect PAL with the *philosophical* tradition of the square of opposition:
  - squares/hexagons arise in a *nontrivial* way in PAL
  - squares/hexagons are *compact representations* of the subtle relations between knowledge and dynamics studied in PAL

# Outline of the talk

# Disclaimer

- dynamic epistemic logic studies any type of epistemic actions (whispering, cheating, private communication etc. . . )
- public announcement logic is a small fragment: studies only 'public announcements'
- for expository reasons, we will focus on PAL
- everything that follows can be generalized to full DEL

# Intuitive idea

- a community $Ag$ of agents
- an outside source $S$ announces that $\varphi$ publicly and truthfully to all agents $i \in Ag$
- explanation:
  - outside source: $S \notin Ag$ ('$S$ is God')
  - truthful: $S$ can only announce truths, falsehoods cannot be announced (true = true *before* the announcement)
  - public: each agent $i \in Ag$ hears the announcement, but $i$ also sees that the others have heard the announcement, and $i$ sees that the others have seen that $i$ has heard the announcement, etc...

# Models

Multi-agent Kripke model $\mathbb{M} = \langle W, \{R_i\}_{i \in Ag}, V \rangle$

- $W$: nonempty set of possible worlds
- $R_i$: equivalence relation on $W$ (for each agent $i \in Ag$)
- $V : Prop \to \wp(W)$: valuation

(Equivalence relations $R_i$, so we will get $S5$-knowledge operators.)

# Updating a model

Take three ingredients:

- a multi-agent Kripke model $\mathbb{M} = \langle W, \{R_i\}_{i \in Ag}, V \rangle$
- a world $w \in W$
- a formula $\varphi \in \mathcal{L}$

If (and only if!) $\mathbb{M}, w \models \varphi$, we define the updated model and world $\mathbb{M}|\varphi, w^\varphi := \langle W^\varphi, \{R_i^\varphi\}_{i \in Ag}, V^\varphi \rangle, w^\varphi$:

- $W^\varphi := \{v \in W \mid \mathbb{M}, v \models \varphi\}$
- $R_i^\varphi := R_i \cap \left(W^\varphi \times W^\varphi\right)$
- $V^\varphi(p) := V(p) \cap W^\varphi$ for any $p \in Prop$
- $w^\varphi := w$

# Formal language

$\mathcal{L}$ is defined as follows

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi \mid [!\varphi]\varphi$$

- $K_i\varphi$: 'agent $i$ knows that $\varphi$'
- $[!\varphi]\psi$: '$\psi$ is true after any public announcement of $\varphi$'
- 'any' might be an empty quantifier here: $\varphi$ might be false and thus not announcable

- dually: $\langle!\varphi\rangle\psi := \neg[!\varphi]\neg\psi$
- '$\varphi$ can be announced, and afterwards $\psi$ is true'

# Semantics

- $\mathbb{M}, w \models p$ iff $w \in V(p)$
- $\mathbb{M}, w \models \neg\varphi$ iff $\mathbb{M}, w \not\models \varphi$
- $\mathbb{M}, w \models \varphi \wedge \psi$ iff $\mathbb{M}, w \models \varphi$ and $\mathbb{M}, w \models \psi$
- $\mathbb{M}, w \models K_i\varphi$ iff $\mathbb{M}, v \models \varphi$ for all $v \in W$ such that $wR_iv$
- $\mathbb{M}, w \models [!\varphi]\psi$ iff if $\mathbb{M}, w \models \varphi$ then $\mathbb{M}|\varphi, w \models \psi$

- $\mathbb{M}, w \models \langle!\varphi\rangle\psi$ iff $\mathbb{M}, w \models \varphi$ and $\mathbb{M}|\varphi, w \models \psi$

We say that PAL $\models \varphi$ iff $\mathbb{M}, w \models \varphi$ for all $\mathbb{M}, w$.

# Some key validities

- truthfulness: PAL $\models \varphi \leftrightarrow \langle !\varphi \rangle \top$
- partiality: PAL $\not\models \langle !\varphi \rangle \top$
- PAL $\not\models [!\varphi]\psi \rightarrow \langle !\varphi \rangle \psi$ (consider $\mathbb{M}, w \not\models \varphi$)
- functionality: PAL $\models \langle !\varphi \rangle \psi \rightarrow [!\varphi]\psi$
- distributivity: PAL $\models [!\varphi](\psi \rightarrow \chi) \rightarrow ([!\varphi]\psi \rightarrow [!\varphi]\chi)$
- necessitation: if PAL $\models \psi$ then also PAL $\models [!\varphi]\psi$

- these are all 'structural' validities
- interaction of knowledge and public announcement: later

# A failed first attempt

- we now start looking for squares of opposition in PAL

- $[!\varphi]$ satisfies distr. and necess. $\Rightarrow$ normal modal operator
- given the modal analysis of knowledge in classical (static) epistemic logic, it is tempting to construct a square like this:

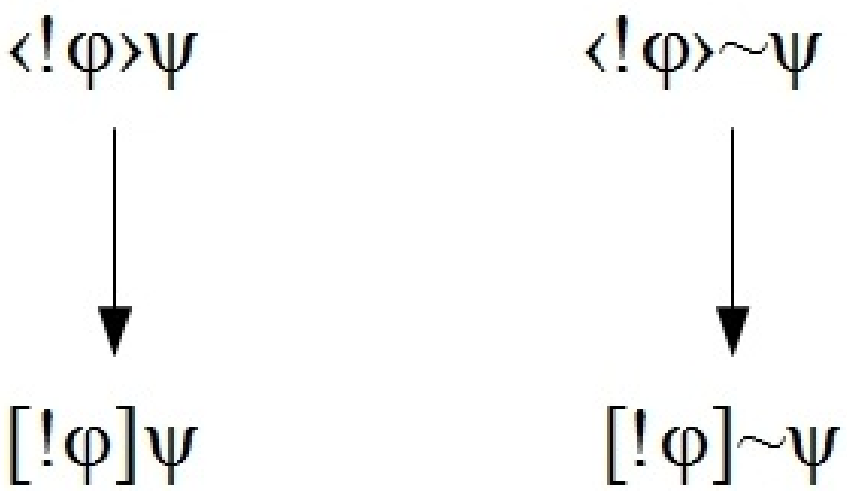
$[!\varphi]\psi$ $\qquad$ $[!\varphi]\sim\psi$

$\langle!\varphi\rangle\psi$ $\qquad$ $\langle!\varphi\rangle\sim\psi$

- but since PAL $\not\models [!\varphi]\psi \rightarrow \langle!\varphi\rangle\psi$, this doesn't work!

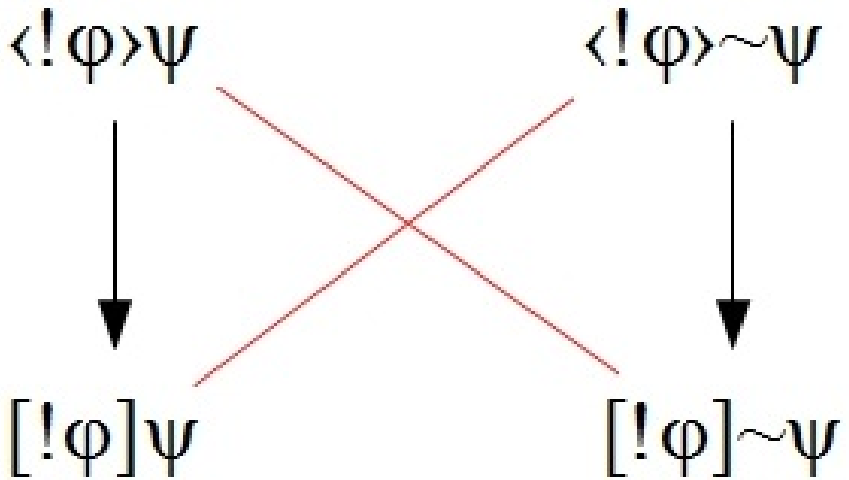
# Building the right square

- but recall that we have functionality: PAL $\models \langle !\varphi\rangle\psi \rightarrow [!\varphi]\psi$, to get subalternation working in 'reversed order':



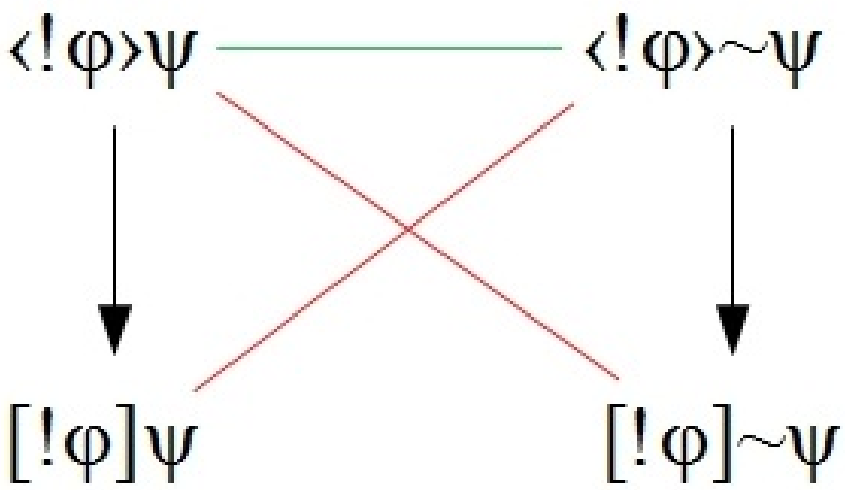
# Building the right square

- since $\langle !\varphi \rangle \psi$ just abbreviates $\neg[!\varphi]\neg\psi$, we get the contradictories:



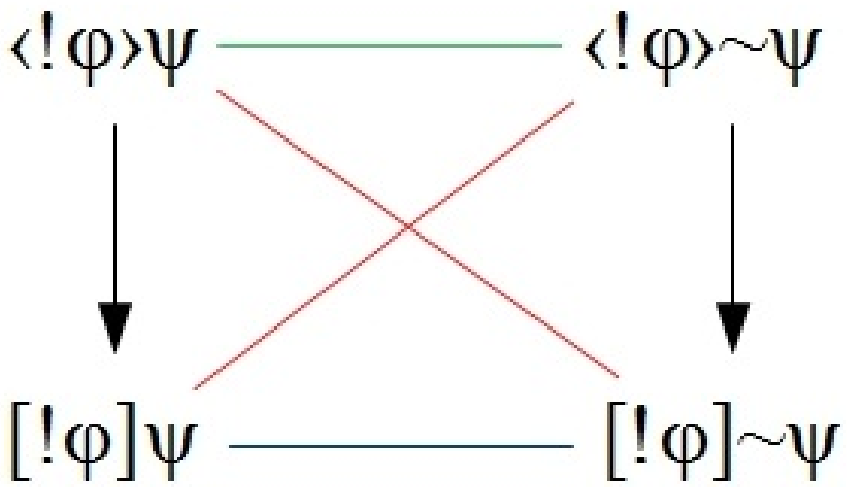
# Building the right square

- $\langle !\varphi \rangle \psi$ and $\langle !\varphi \rangle \neg \psi$ can both be false (if $\varphi$ is false), but they cannot both be true (check the semantics!), so they are contraries:



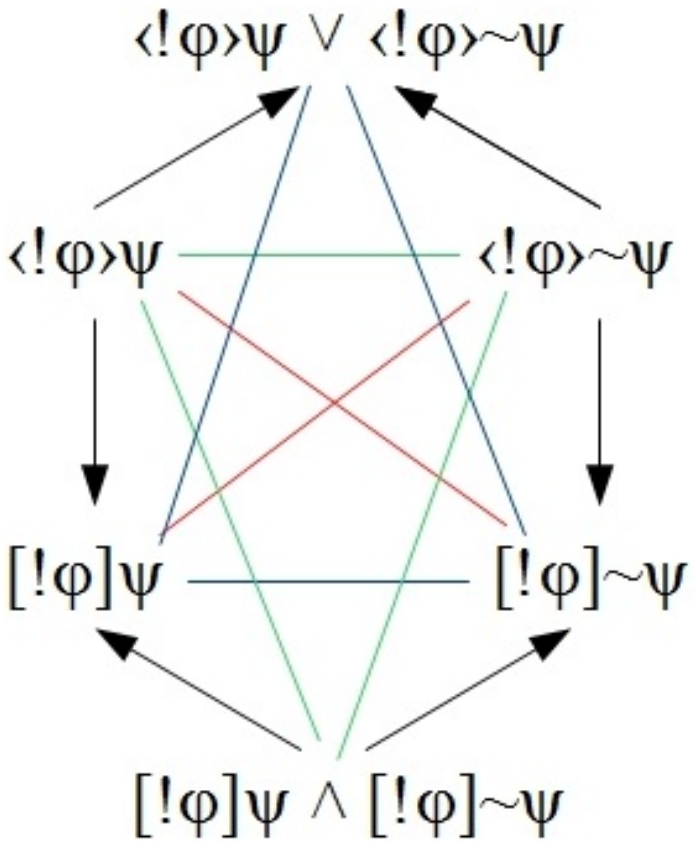
# Building the right square

- $[!\varphi]\psi$ and $[!\varphi]\neg\psi$ can both be true (if $\varphi$ is false), but they cannot both be false, so they are subcontraries:



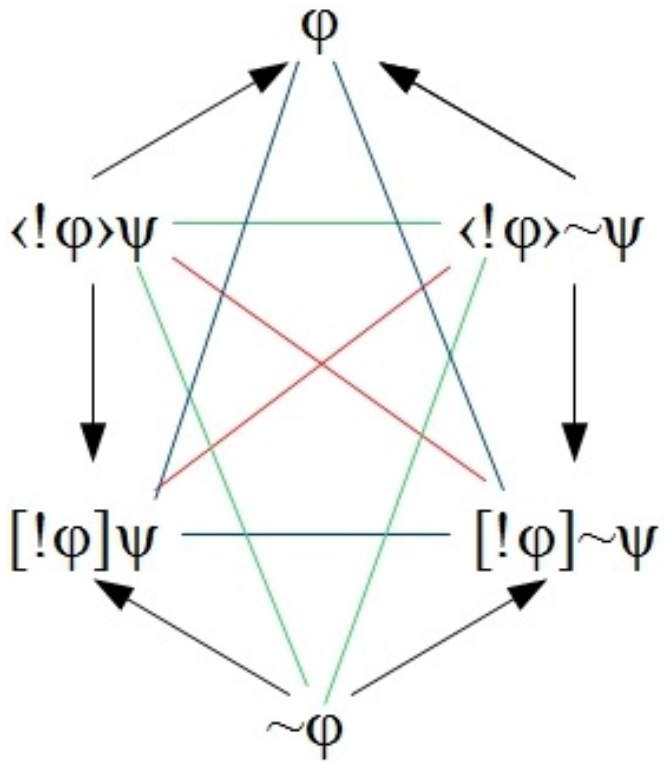
# Building the right square

- using well-known techniques from Sesmat and Blanché, we turn the square into a hexagon:



# Building the right square

- but $\langle !\varphi\rangle\psi \vee \langle !\varphi\rangle\neg\psi$ just says that $\varphi$ is announcable at all, and thus true; similarly $[!\varphi]\psi \wedge [!\varphi]\neg\psi$ says that $\varphi$ is not announcable, and thus false:



# Summary

- we have constructed a square (hexagon) for public announcements
- in most well-known squares, implications go from the universal notion ($\forall, \Box, K, O$) to the existential notion ($\exists, \Diamond, \hat{K}$, P)
- in our square the implications are reversed!
- other relations (contradiction, contr., subcontr.) still ok
- $\Rightarrow$ the 'reversed' square is really natural!

# Some more validities

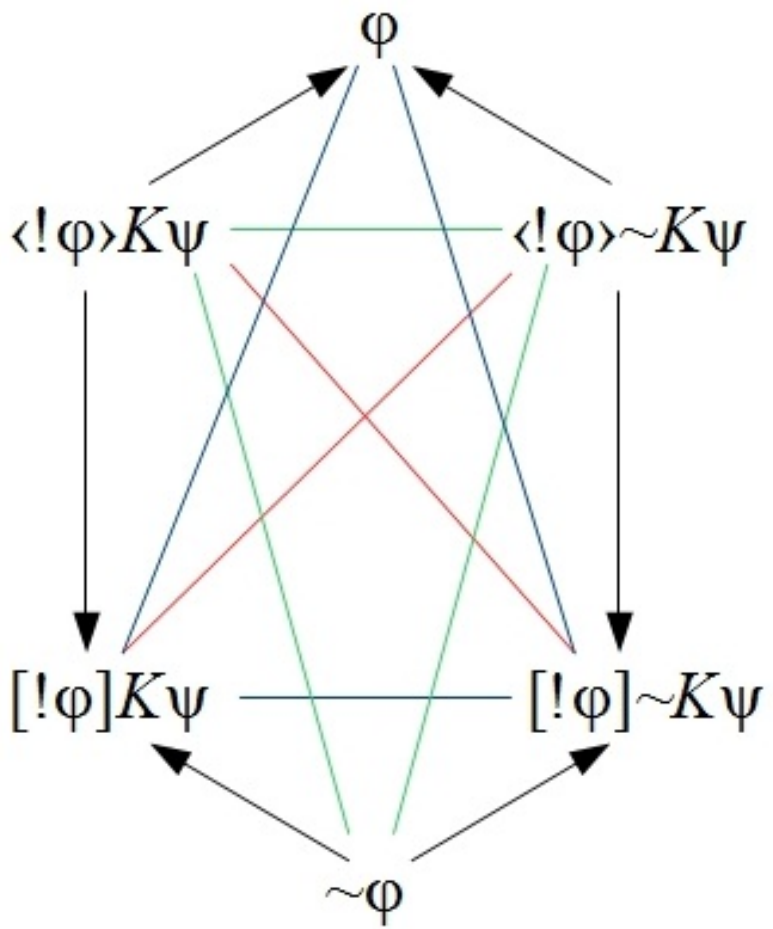
We will now extend the hexagon constructed above with knowledge

First some validities about the interaction between knowledge and public announcement:

1. PAL $\models \langle !\varphi \rangle K_i \psi \rightarrow K_i [!\varphi] \psi$
2. PAL $\models K_i [!\varphi] \psi \rightarrow [!\varphi] K_i \psi$
3. PAL $\models \langle !\varphi \rangle \neg K_i \psi \rightarrow \neg K_i [!\varphi] \psi$ (contrapositive of 2)
4. PAL $\models \neg K_i [!\varphi] \psi \rightarrow [!\varphi] \neg K_i \psi$ (contrapositive of 1)

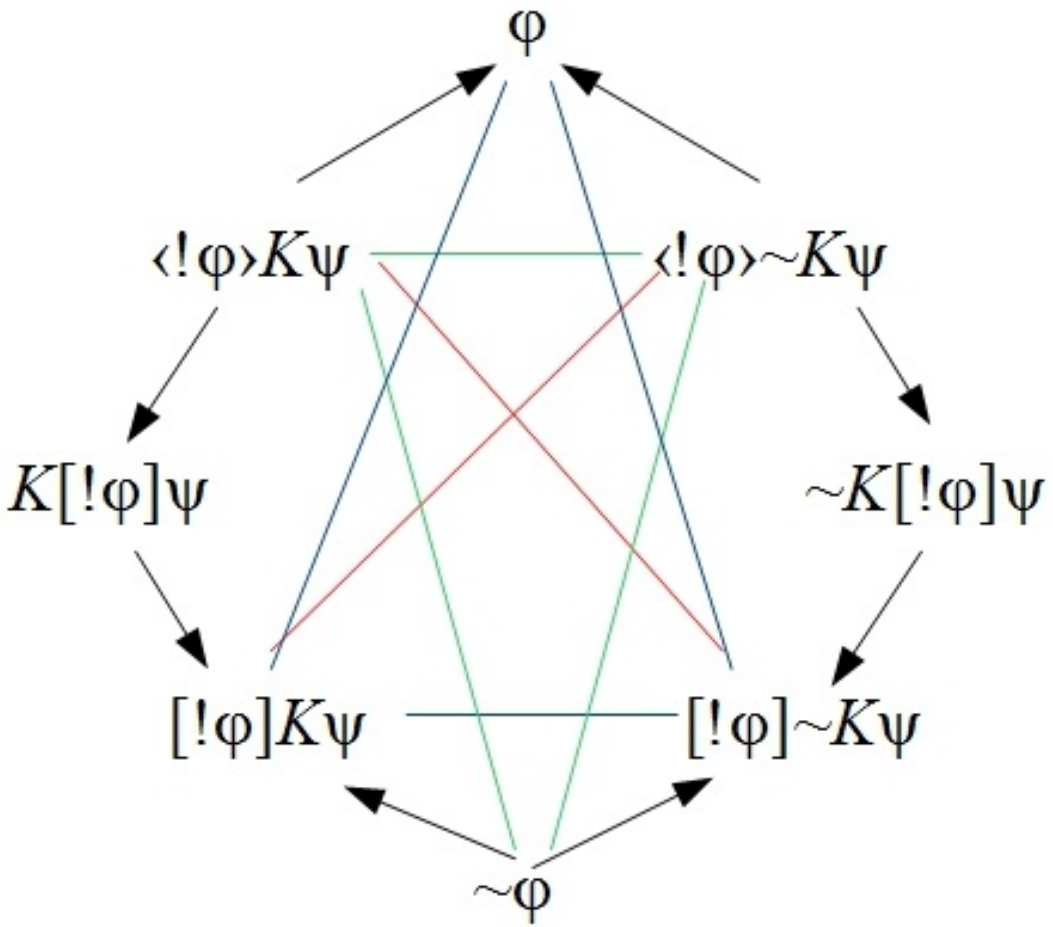
# Building the second square

- take the square/hexagon constructed above, but replace $\psi$ with $K\psi$ (we drop agent indices):



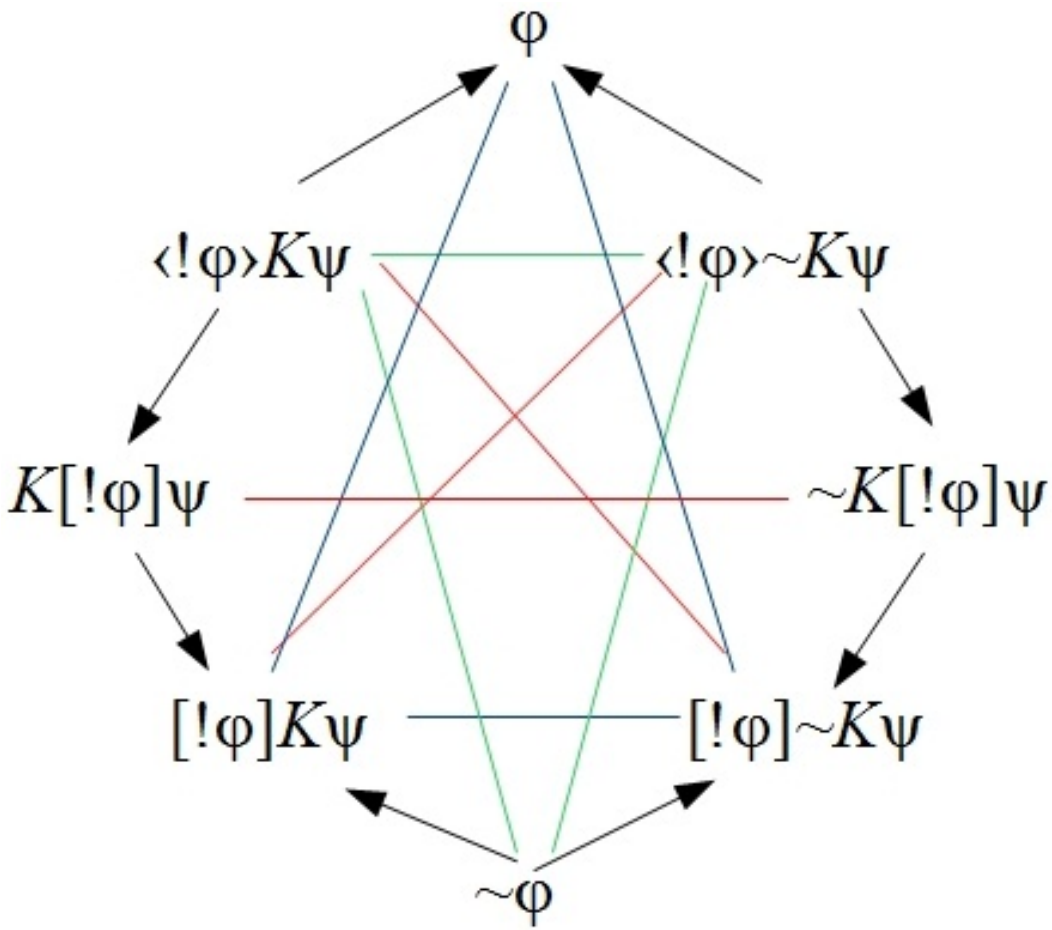
# Building the second square

- using the four validities above, we extend the hexagon:



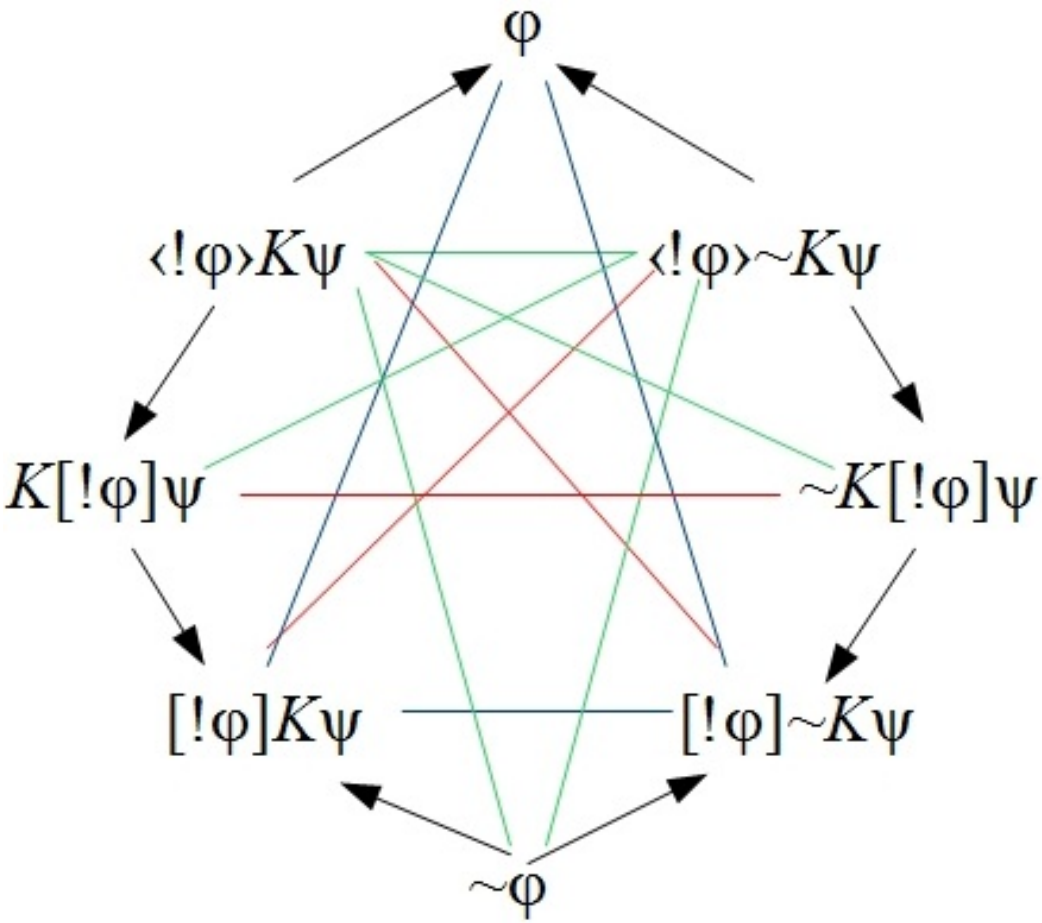
# Building the second square

- obviously $K[!\varphi]\psi$ and $\neg K[!\varphi]\psi$ are contradictory:



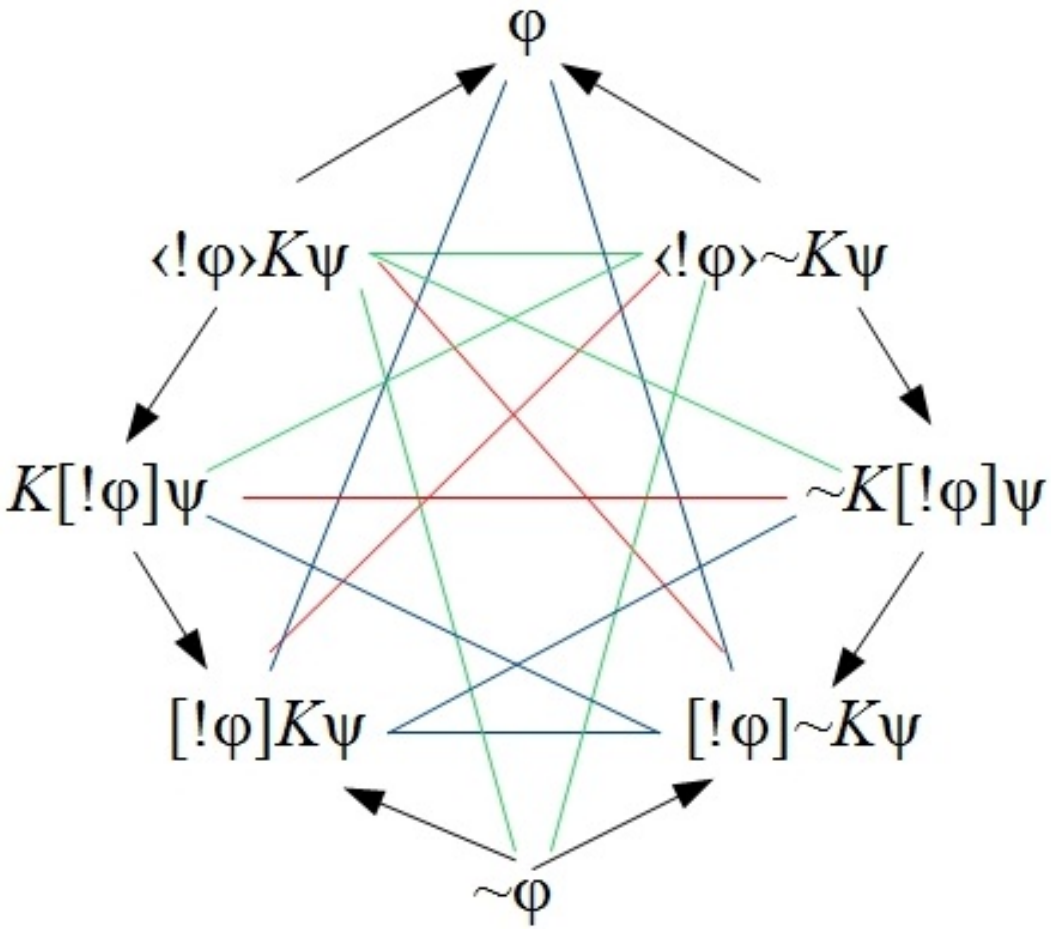
# Building the second square

- two more contrariety relations (can both be false if $\varphi$ false):



# Building the second square

- two more subcontrariety relations (can both be true if $\varphi$ false):



# Summary

- we have extended the first (reversed) hexagon with knowledge
- by adding two more formulas, we obtained:
  - four more implication (altern/subaltern) relations
  - two more contradiction relations
  - two more contrariety relations
  - two more subcontrariety relations
- PAL provides a detailed account of the subtle interactions between knowledge and public announcement
- the (extended) hexagon is a compact representation of this account

# Conclusion and future work

- aim: connect PAL (DEL) with the rich philosophical tradition of the square of opposition
- our squares/hexagons are:
  - *interesting*: they arise in a nontrivial way
  - *useful*: compact representation of a lot of information

- future work: study this at higher level of abstraction
  - partial functionality is essential for the 'reversed' squares (also in generalization to DEL)
  - drop the epistemic perspective altogether
  - reversed squares for dynamic logic of (deterministic) computer programs

# Thank you!

Handout available at
`http://perswww.kuleuven.be/lorenz_demey`